

ALGEBRAIC GAUSS-MANIN SYSTEMS AND BRIESKORN MODULES

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ABSTRACT. We study the algebraic Gauss-Manin system and the algebraic Brieskorn module associated to a polynomial mapping with isolated singularities. Since the algebraic Gauss-Manin system does not contain any information on the cohomology of singular fibers, we first construct a non quasi-coherent sheaf which gives the cohomology of every fiber. Then we study the algebraic Brieskorn module, and show that its position in the algebraic Gauss-Manin system is determined by a natural map to quotients of local analytic Gauss-Manin systems, and its pole part by the vanishing cycles at infinity, comparing it with the Deligne extension. This implies for example a formula for the determinant of periods. In the two-dimensional case we can describe the global structure of the algebraic Gauss-Manin system rather explicitly.

Introduction

Let $f : X \rightarrow S$ be a morphism of smooth complex algebraic varieties. The Gauss-Manin systems \mathcal{G}_f^i of f are defined to be the (cohomological) direct images of \mathcal{O}_X as algebraic left \mathcal{D} -Modules. They are regular holonomic \mathcal{D}_S -Modules, and correspond by the de Rham functor DR to the (perverse) higher direct images of the constant sheaf. In particular, they give the cohomology with compact support of each fiber, but not the cohomology of singular fibers unless f is proper, because the stalk of the higher direct image does not coincide with the cohomology of the fiber due to the vanishing cycles at infinity. Furthermore the cohomologies of the fibers do not form a constructible sheaf in a natural way (see (2.6)), and it is not easy to construct a quasi-coherent sheaf on S which is generically coherent, and contains the information on the cohomology of every fiber.

Assume for simplicity $X = \mathbb{A}^n, S = \mathbb{A}^1$ with $n \geq 2$. Then it is easy to show that

$$\mathrm{DR}_S(\mathcal{G}_f^i) = (R^{n+i-1}f_*\mathbb{C}_X)[1]$$

in the category of perverse sheaves $\mathrm{Perv}(S, \mathbb{C})$ (see [3]). Let $\mathcal{G}_f = \mathcal{G}_f^0$, $G_f = \Gamma(S, \mathcal{G}_f)$, and $L = R^{n-1}f_*\mathbb{C}_X$. Note that

$$\mathcal{G}_f^i = 0 \quad \text{for } i \neq 1 - n, 0,$$

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if f has at most isolated singularities including at infinity [31] (or, more generally, if the support of the vanishing cycles including at infinity is discrete). See (1.3).

For a variety Y of pure dimension m in general, we have the trace morphism $\text{Tr} : H_c^{2m}(Y, \mathbb{C}) \rightarrow \mathbb{C}$, and $\tilde{H}_c^i(Y, \mathbb{C})$ is defined as $\text{Ker}(\text{Tr} : H_c^{2m}(Y, \mathbb{C}) \rightarrow \mathbb{C})$ for $i = 2m$, and $H_c^i(Y, \mathbb{C})$ otherwise.

For $c \in \mathbb{C}$, let $X_c = f^{-1}(c)$. If f has at most isolated singularities including at infinity, then $\tilde{H}_c^i(X_c, \mathbb{C}) = 0$ for $i \neq n-1, n$, and we can easily show canonical isomorphisms

$$\text{Ker}(t - c|G_f) = \tilde{H}_c^n(X_c, \mathbb{C})^*, \quad \text{Coker}(t - c|G_f) = H_c^{n-1}(X_c, \mathbb{C})^*,$$

where $*$ denotes the dual vector space. See (1.2–3).

From now on, we assume that f has at most isolated singularities. Let $\Omega^i = \Gamma(X, \Omega_X^i)$. Then there exists a natural morphism $\Omega^n \rightarrow G_f$, and its kernel is $df \wedge d\Omega^{n-2}$. So we have a $\mathbb{C}[t]$ -submodule

$$G_f^{(0)} = \Omega^n / df \wedge d\Omega^{n-2} \subset G_f,$$

which is called the *algebraic Brieskorn module* of f . Let $\mathcal{G}_f^{(0)}$ denote the quasi-coherent sheaf corresponding to $G_f^{(0)}$, and $\mathcal{G}_f^{(0), \text{an}}$ the associated analytic sheaf.

For $x \in X$, let $\mathcal{G}_{f,x}^{(0)}$ denote the local analytic Brieskorn lattice $\Omega_{X^{\text{an}},x}^n / df \wedge d\Omega_{X^{\text{an}},x}^{n-2}$ (see [7]), and $\mathcal{G}_{f,x}$ the local analytic Gauss-Manin system [21] which is the localization of $\mathcal{G}_f^{(0)}$ by ∂_t^{-1} . (They vanish unless $x \in \text{Sing } f$.) Then for $x \in X_c$, we have the restriction morphism $\mathcal{G}_{f,c}^{(0), \text{an}} \rightarrow \mathcal{G}_{f,x}^{(0)}$. We define

$$\mathcal{L}_{f,c} = \text{Ker}(\mathcal{G}_{f,c}^{(0), \text{an}} \rightarrow \bigoplus_{f(x)=c} \mathcal{G}_{f,x}^{(0)}).$$

0.1. Theorem. *If f has at most isolated singularities including at infinity, we have natural isomorphisms*

$$\text{Ker}(t - c|\mathcal{L}_{f,c}) = \tilde{H}^{n-2}(X_c, \mathbb{C}), \quad \text{Coker}(t - c|\mathcal{L}_{f,c}) = H^{n-1}(X_c, \mathbb{C}),$$

and $\tilde{H}^i(X_c, \mathbb{C}) = 0$ for $i \neq n-2, n-1$. See (2.4).

But the $\mathcal{L}_{f,c}$ do not form a quasi-coherent sheaf on S^{an} , and $\mathcal{G}_{f,c}^{(0), \text{an}}$ in the definition of $\mathcal{L}_{f,c}$ cannot be replaced by $\mathcal{G}_{f,c}^{(0)}$ or $G_f^{(0)}$. (Indeed, the image of the natural morphism $G_f^{(0)} \rightarrow \mathcal{G}_{f,x}^{(0)}$ is not a $\mathbb{C}[t]$ -module of rank μ_x in general, where μ_x denote the Milnor number of f at $x \in \text{Sing } f$.) From (0.1) we get

0.2. Corollary. With the assumption of (0.1) we have

$$\begin{aligned} \dim \text{Ker}(t - c|G_f^{(0)}) &= \dim \tilde{H}^{n-2}(X_c, \mathbb{C}), \\ \dim \text{Coker}(t - c|G_f^{(0)}) &= \dim H^{n-1}(X_c, \mathbb{C}) + \sum_{f(x)=c} \mu_x. \end{aligned}$$

See also (3.6) for the relation with other invariants.

In the local analytic case, the Brieskorn lattice $\mathcal{G}_{f,x}^{(0)}$ gives the Hodge numbers of the local analytic Milnor fiber [32] by [35] (see also [22], [27], [29], etc.), and contains the information on the analytic structure of (f, x) in the local moduli space (see for example [28]). However, concerning the global algebraic structure of f , the algebraic Brieskorn module $G_f^{(0)}$ does not contain any more than the information on the local analytic structure of (f, x) at $x \in \text{Sing } f$ (although it gives the sums of the Hodge numbers of local Milnor fibers, see (3.5)), because $G_f^{(0)}$ is determined only by G_f together with the composition of natural morphisms $\mathcal{G}_{f,c}^{\text{an}} \rightarrow \mathcal{G}_{f,x} \rightarrow \mathcal{G}_{f,x}/\mathcal{G}_{f,x}^{(0)}$ for $x \in \text{Sing } f$ due to the following :

0.3. Theorem.

$$G_f^{(0)} = \text{Ker}\left(G_f \rightarrow \bigoplus_{x \in \text{Sing } f} \mathcal{G}_{f,x}/\mathcal{G}_{f,x}^{(0)}\right).$$

(This is a part of Theorem (0.5) below.) Note that G_f is determined by the constructible sheaf $L(:= R^{n-1}f_*\mathbb{C}_X)$ by using the Riemann-Hilbert correspondence (see for example [4]), and the morphism $\mathcal{G}_{f,c}^{\text{an}} \rightarrow \mathcal{G}_{f,x}$ for $x \in X_c$ is determined also topologically by using the restriction to the local Milnor fibration. It is relatively easy to describe $\mathcal{G}_f^{\text{an}}$ at least locally, using the local classification of regular holonomic \mathcal{D} -module of one variable [5], [6] (see also [26, 1.3]).

From (0.3) we also get

0.4. Corollary.

$$\mathcal{L}_{f,c} = \text{Ker}\left(\mathcal{G}_{f,c}^{\text{an}} \rightarrow \bigoplus_{f(x)=c} \mathcal{G}_{f,x}\right).$$

In particular, $\mathcal{L}_{f,c}$ is actually a $D_{S,c}^{\text{an}}$ -module (on which the action of ∂_t is surjective). The corresponding constructible sheaf defined on a neighborhood S' of c is given by $R^{n-1}f_*\tilde{j}!\mathbb{C}_{X \setminus B}|_{S'}$ where $B = \bigcup_{x \in \text{Sing } f} B_x$ with B_x a sufficiently small ball with center x , and $\tilde{j} : X \setminus B \rightarrow X$ denotes the inclusion.

In general, $G_f^{(0)}$ is not finitely generated over $\mathbb{C}[t]$. As a corollary of Sabbah's results [23], $G_f^{(0)}$ is a finite $\mathbb{C}[t]$ -module if and only if f has a certain good property at infinity (i.e. if f is cohomologically tame in his sense). See (3.3). Let U be a dense open subvariety of S such that $L|_U$ is a local system (i.e. $\mathcal{G}_f|_U$ is a locally free \mathcal{O}_U -Module of finite rank). Put $\Delta = S \setminus U$. Let $\mathcal{G}_f^{>0}$ be the Deligne extension of $\mathcal{G}_f|_U$ such that the eigenvalues of the residues of the connection are contained in $(-1, 0]$. See [9]. (The shift of the index comes from the normalization of the exponents in [25].) Put $G_f^{>0} = \Gamma(S, \mathcal{G}_f^{>0})$. Then $G_f^{>0}$ is always a free $\mathbb{C}[t]$ -module of finite type. We will see as a corollary of (0.5) below that $G_f^{(0)}$ is finite over $\mathbb{C}[t]$ if and only if it is contained in $G_f^{>0}$. Let

$$G_f^{(0), >0} = G_f^{(0)} \cap G_f^{>0}.$$

This is called the *Brieskorn-Deligne lattice*. It is a free $\mathbb{C}[t]$ -module of finite rank, and generates G_f over $\Gamma(S, \mathcal{D}_S)$, because \mathcal{G}_f has no nontrivial quotient with discrete support. See (1.3). For $x \in \text{Sing } f$, we can define the Deligne extension $\mathcal{G}_{f,x}^{>0}$ (which is contained in $\mathcal{G}_{f,x}$) similarly. Then

0.5. Theorem. *We have the following commutative diagram of exact sequences of $\mathbb{C}[t]$ -modules. See (3.2).*

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & G_f^{(0), >0} & \longrightarrow & G_f^{>0} & \longrightarrow & \bigoplus_{x \in \text{Sing } f} \mathcal{G}_{f,x}^{>0} / \mathcal{G}_{f,x}^{(0)} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & G_f^{(0)} & \longrightarrow & G_f & \longrightarrow & \bigoplus_{x \in \text{Sing } f} \mathcal{G}_{f,x} / \mathcal{G}_{f,x}^{(0)} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \bigoplus_{c \in \Delta} P_c(E'_c, T) & \longrightarrow & \bigoplus_{c \in \Delta} P_c(E_c, T) & \longrightarrow & \bigoplus_{x \in \text{Sing } f} P_{f(x)}(E''_x, T) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

See (3.1) for $P_c(E'_c, T)$, etc., where c and $t - c$ are denoted by s and t_s respectively. Note that $P_c(E'_c, T)$ is isomorphic to $E'_c \otimes_{\mathbb{C}} \mathbb{C}[\frac{1}{t-c}] \frac{1}{t-c}$ as a $\mathbb{C}[t]$ -module (where $\mathbb{C}[\frac{1}{t-c}] \frac{1}{t-c} := \mathbb{C}[t, \frac{1}{t-c}] / \mathbb{C}[t]$), and is called the pole part of $G_f^{(0)}$ at c . So every vanishing cycle at infinity contributes to the pole part of $G_f^{(0)}$, since E'_c is the space of vanishing cycles at infinity. For the proof of (0.5), it is not necessary to use the theory of the direct image of the filtration V of Kashiwara [18] and Malgrange [20] as in [23], [24], [27], because it is actually enough to use a much easier theory of the filtration V in the one variable case which is closely related to the Deligne extension [9] and also to Varchenko's theory [35], [36].

Theorem (0.5) describes the structure of $G_f^{(0)}$ by comparing it with the Deligne extension $G_f^{>0}$. The first row means that $G_f^{(0), >0}$ coincides with $G_f^{>0}$ up to torsion, because we have for $x \in \text{Sing } X_c$

$$\mathcal{G}_{f,x}^{>0} / \mathcal{G}_{f,x}^{(0)} = \sum_{1 \leq j \leq \mu_x} \mathbb{C}[t] / (t - c)^{k_j} \mathbb{C}[t]$$

as $\mathbb{C}[t]$ -modules. Here k_j are nonnegative integers such that $0 < \alpha_j - k_j \leq 1$ with α_j ($1 \leq j \leq \mu_x$) the exponents of the Brieskorn lattice. See [26]. Let μ_x^1 be the dimension of the unipotent monodromy part of the vanishing cohomology of f at x (i.e. the number of the exponents which are integers.) Then from the first row of (0.5) together with the symmetry of the exponents, we get

0.6. Corollary.

$$\dim \mathcal{G}_{f,c}^{>0} / \mathcal{G}_{f,c}^{(0), >0} = \sum_{f(x)=c} \frac{1}{2} ((n-1)\mu_x - \mu_x^1).$$

In particular, $G_f^{(0),>0} = G_f^{>0}$ if and only if $n = 2$ and every singular point of f on X is an ordinary double point (i.e., a node), because the minimal exponent has multiplicity one.

Let $G_f^{(-1)} = df \wedge \Omega^{n-1}/df \wedge d\Omega^{n-2} (= \partial_t^{-1} G_f^{(0)}) \subset G_f^{(0)}$ (similarly for $\mathcal{G}_{f,x}^{(-1)}$), and $G_f^{(-1),>0} = G_f^{(-1)} \cap G_f^{>0}$. Then (0.5) holds with $G_f^{(0)}$, $G_f^{(0),>0}$, $\mathcal{G}_{f,x}^{(0)}$ replaced by $G_f^{(-1)}$, $G_f^{(-1),>0}$, $\mathcal{G}_{f,x}^{(-1)}$.

Let $\{\omega_i\}$, $\{df \wedge \eta_i\}$ be $\mathbb{C}[t]$ -bases of $G_f^{(0),>0}$, $G_f^{(-1),>0}$, and $\{\gamma_j(c)\}$ a basis of horizontal (multivalued) sections of $\prod_{c \in U} H_{n-1}(X_c, \mathbb{Z})/\text{torsion}$. Then the squares of the determinants of the period matrices

$$\det \left(\int_{\gamma_j(t)} \text{res} \frac{\omega_i}{f-t} \right)^2, \quad \det \left(\int_{\gamma_j(t)} \eta_i \right)^2 \quad \text{for } t \in U$$

are holomorphic functions on U , and are independent of the choice of the bases up to constant multiples. Here $\text{res} \frac{\omega_i}{f-t}$ denotes the Poincaré residue. (See for example, [7, 1.5]). For $c \in \Delta := S \setminus U$, let $\nu_c^{\neq 1}$ be the dimension of the nonunipotent monodromy part of E'_c , and put

$$m_c = -\nu_c^{\neq 1} + \sum_{f(x)=c} (n-2)\mu_x, \quad m'_c = -\nu_c^{\neq 1} + \sum_{f(x)=c} n\mu_x.$$

Then from (0.5) we can deduce (see (3.4)) :

0.7. Corollary.

$$\det \left(\int_{\gamma_j(t)} \text{res} \frac{\omega_i}{f-t} \right)^2 = \text{const} \prod_{c \in \Delta} (t-c)^{m_c},$$

$$\det \left(\int_{\gamma_j(t)} \eta_i \right)^2 = \text{const} \prod_{c \in \Delta} (t-c)^{m'_c}.$$

This generalizes [16, 2.2] where $n = 2$ and f is a semiweighted homogeneous polynomial in the sense of loc. cit. (in particular, f is tame and $\nu = 0$). The corresponding assertion for the local analytic Brieskorn lattice is well-known. See [36].

Since the algebraic Brieskorn module is determined by the algebraic Gauss-Manin system G_f together with the map to the quotients of the local analytic Gauss-Manin systems, we consider the problem of determining the global structure of the Gauss-Manin system. We treat rather the perverse sheaf

$$L[1] := (R^{n-1} f_* \mathbb{C}_X)[1]$$

corresponding to \mathcal{G}_f by the Riemann-Hilbert correspondence, because it is easier to handle. In the case $n = 2$, we can describe it rather explicitly by using the relative version of Deligne's weight spectral sequence [11] (see [24], [25]).

Let $\bar{f} : \bar{X} \rightarrow S$ be a relative compactification of f such that \bar{X} is smooth and $\bar{L} := R^1 \bar{f}_* \mathbb{C}_{\bar{X}}|_U$ is a local system (shrinking U if necessary). Let g be the genus of the generic

fiber of \bar{f} such that $\text{rank } \bar{L} = 2g$. Let $j : U \rightarrow X$ be the inclusion morphism. Then $(j_*\bar{L})[1]$ is the intersection complex $\text{IC}_S \bar{L}$ (i.e. the intermediate direct image of $\bar{L}[1]$ in the sense of [3]). See also [37]. Let h be the number of horizontal irreducible components of $\bar{X} \setminus X$, and n_c the number of irreducible components of X_c . Let W be the weight filtration on $L[1]$ in the theory of mixed Hodge Modules [24] [25] (this can also be obtained by using a mod p reduction as in [3]). Then, using the weight spectral sequence, we can easily show the following :

0.8. Proposition. *With the above notation and assumption, $\text{Gr}_k^W(L[1]) = 0$ for $k \neq 2, 3$, and*

$$\begin{aligned} \text{Gr}_2^W(L[1]) &= \left(\bigoplus_{c \in \Delta} \left(\bigoplus^{n_c-1} \mathbb{C}_{\{c\}} \right) \right) \oplus (j_*\bar{L})[1], \\ \text{Gr}_3^W(L[1]) &= \left(\bigoplus^{h-1} \mathbb{C}_S[1] \right) \oplus \left(\bigoplus_i (j_*L'_i)[1] \right), \end{aligned}$$

where the L'_i are non constant irreducible local systems on U .

(The proof is more or less standard, and is left to the reader. Indeed, the vanishing of $\text{Gr}_4^W(L[1])$ follows from (1.3), and the multiplicity $n_c - 1$ of $\mathbb{C}_{\{c\}}$ is determined by using for example (1.2).) By the long exact sequence associated with the filtration W on $L[1]$, we get a refinement of Kaliman's inequality [17]:

0.9. Corollary.

$$h - 1 = \sum_{c \in \Delta} (n_c - 1) + \dim H^1(S^{\text{an}}, j_*\bar{L}).$$

Passing to the corresponding regular holonomic \mathcal{D} -modules, we get submodules M' , M'' of \mathcal{G}_f with a short exact sequence

$$0 \rightarrow M' \bigoplus M'' \rightarrow \mathcal{G}_f \rightarrow \tilde{M} \rightarrow 0,$$

such that $M' = \bigoplus_c \left(\bigoplus^{n_c-1} \mathcal{D}_S / \mathcal{D}_S(t-c)\partial_t \right)$, $\tilde{M} = \bigoplus_i \tilde{M}_i$ with $\text{DR}_S(\tilde{M}_i) = (j_*L'_i)[1]$. As to M'' , we have a short exact sequence

$$0 \rightarrow M_S(\bar{L}) \rightarrow M'' \rightarrow \bigoplus^{h''} \mathcal{O}_S \rightarrow 0,$$

such that $\text{DR}_S(M_S(\bar{L})) = (j_*\bar{L})[1]$, where $h'' = \dim H^1(S^{\text{an}}, j_*\bar{L})$. So the description of \mathcal{G}_f is essentially reduced to the problem of extension. For example, if $r = 0$ (i.e. $f : Y_j \rightarrow S$ is bijective for any irreducible component Y_j of $\bar{X} \setminus X$ such that $\bar{f}(Y_j) = S$), then we have $\mathcal{G}_f = M' \oplus M''$. Note that $r = 0$ by [15] if $n = 2$ and f is cohomologically tame [23] (see also [2]).

Note that (0.8) implies a nontrivial assertion on the local system $L|_U$. In terms of the corresponding \mathcal{D} -modules, let ${}_{\text{tor}}\mathcal{G}_f$ be the maximal \mathcal{D}_S -submodule of \mathcal{G}_f whose support is contained in Δ . Then we have

$${}_{\text{tor}}\mathcal{G}_f = \bigoplus_c \left(\bigoplus^{n_c-1} \mathcal{D}_S / \mathcal{D}_S(t-c) \right) \subset M', \quad \text{DR}_S({}_{\text{tor}}\mathcal{G}_f) = \bigoplus_c \left(\bigoplus^{n_c-1} \mathbb{C}_{\{c\}} \right),$$

and $M'/_{\text{tor}}\mathcal{G}_f (= \bigoplus^{h'} \mathcal{O}_S)$ is a direct factor of $\mathcal{G}_f/_{\text{tor}}\mathcal{G}_f$, where $h' = \sum_c (n_c - 1)$. (This follows from the semisimplicity of $\text{Gr}_3^W \mathcal{G}_f$.) It implies a result of Bailly-Maitre [2] that the maximal constant subsheaf of $L|_U$ has rank h' (using [14]), and is a direct factor of $L|_U$.

In Sect. 1, we introduce the algebraic Gauss-Manin system, and study the action of $t - c$ on it. In Sect. 2, we define a filtration on the algebraic Gauss-Manin system which gives the algebraic Brieskorn module, and prove (0.1–2) in a more general situation. Then Theorem (0.5) and its corollaries are proved in Sect. 3.

Convention. In this paper, algebraic variety means a separated scheme of finite type over \mathbb{C} . For a variety X and a morphism f , we denote $\mathbb{C}_{X^{\text{an}}}$ by \mathbb{C}_X , and f^{an} by f to simplify the notation, where X^{an} is the associated analytic space. Similarly, $H^i(X, \mathbb{C})$ and $H_c^i(X, \mathbb{C})$ denote respectively $H^i(X^{\text{an}}, \mathbb{C})$ and $H_c^i(X^{\text{an}}, \mathbb{C})$. A point of a variety means always a closed point, and $x \in X$ means $x \in X(\mathbb{C})$.

We denote the nearby and vanishing cycle functors [10] by ψ , φ , and $\psi[-1]$, $\varphi[-1]$ by ${}^p\psi$, ${}^p\varphi$, because these preserve perverse sheaves.

1. Algebraic Gauss-Manin systems

1.1. Let $f : X \rightarrow S$ be a morphism of smooth algebraic varieties. Let $n = \dim X$. We assume in this paper $\dim S = 1$, f is not constant and X, S are connected. The Gauss-Manin systems \mathcal{G}_f^i are defined to be the cohomology sheaves $\mathcal{H}^i \mathcal{K}_f$ of the direct image $\mathcal{K}_f := f_+(\mathcal{O}_X)$ of \mathcal{O}_X as algebraic left \mathcal{D} -Modules. We have

$$(1.1.1) \quad \text{DR}_S(\mathcal{G}_f^i) = {}^p R^{i+n} f_* \mathbb{C}_X,$$

where ${}^p R^i f_* \mathbb{C}_X = {}^p \mathcal{H}^i(\mathbf{R}f_* \mathbb{C}_X)$ with ${}^p \mathcal{H}^i : D_c^b(S, \mathbb{C}) \rightarrow \text{Perv}(S, \mathbb{C})$ the perverse cohomology functor in [3].

By assumption, we have a natural injective morphism $\mathcal{O}_S \rightarrow \mathcal{G}_f^{1-n}$. We define the reduced Gauss-Manin systems $\tilde{\mathcal{G}}_f^i$ by

$$\tilde{\mathcal{G}}_f^i = \begin{cases} \text{Coker}(\mathcal{O}_S \rightarrow \mathcal{G}_f^{1-n}) & \text{for } i = 1 - n, \\ \mathcal{G}_f^i & \text{otherwise.} \end{cases}$$

Let $s \in S$ with the natural inclusion $i_s : \{s\} \rightarrow S$. We choose and fix a local coordinate t_s around s such that $\{s\} = \{t_s = 0\}$. For a complex of \mathcal{O}_S -Modules (or $\mathcal{O}_{S^{\text{an}}, s}$ -modules) M , we define

$$(1.1.2) \quad i_s^! M = \text{Cone}(-t_s : M_s \rightarrow M_s)[-1].$$

1.2. Proposition. *We have short exact sequences*

$$(1.2.1) \quad 0 \rightarrow H^1 i_s^! \tilde{\mathcal{G}}_f^{i-1} \rightarrow \tilde{H}_c^{n-i}(X_s, \mathbb{C})^* \rightarrow H^0 i_s^! \tilde{\mathcal{G}}_f^i \rightarrow 0,$$

where $\tilde{H}_c^{n-i}(X_s, \mathbb{C})$ is as in the introduction, and $*$ denotes the dual vector space.

Proof. Let $f_+(\mathcal{O}_X)^\sim = C(\mathcal{O}_S \rightarrow f_+(\mathcal{O}_X))$. Then $\mathcal{H}^i f_+(\mathcal{O}_X)^\sim = \tilde{\mathcal{G}}_f^i$, and we have a spectral sequence

$$E_2^{p,q} = H^p i_s^! \tilde{\mathcal{G}}_f^q \Rightarrow H^{p+q} i_s^! f_+(\mathcal{O}_X)^\sim,$$

degenerating at E_2 (because $E_2^{p,q} = 0$ for $p \neq 0, 1$). So it is enough to show

$$H^i i_s^! f_+(\mathcal{O}_X)^\sim = \tilde{H}_c^{n-i}(X_s, \mathbb{C})^*.$$

Then, using the distinguished triangle $\rightarrow \mathcal{O}_S \rightarrow f_+(\mathcal{O}_X) \rightarrow f_+(\mathcal{O}_X)^\sim \rightarrow$ together with $i_s^! \mathcal{O}_S = \mathbb{C}[-1]$, we can reduce the assertion to

$$H^i i_s^! f_+(\mathcal{O}_X) = H_c^{n-i}(X_s, \mathbb{C})^*.$$

Since the de Rham functor DR commutes with $i_s^!$ and the direct images, we have

$$i_s^! f_+(\mathcal{O}_X) = \mathrm{DR}_{\{s\}}(i_s^! f_+(\mathcal{O}_X)) = i_s^! \mathbf{R}f_* \mathrm{DR}_S(\mathcal{O}_X) = i_s^! \mathbf{R}f_*(\mathbb{C}_X[n]),$$

because $\mathrm{DR}_{\{s\}} = id$ and $\mathrm{DR}_S(\mathcal{O}_X) = \mathbb{C}_X[n]$. Since $\mathbb{C}_X[n]$ is self dual (i.e., $\mathbb{D}(\mathbb{C}_X[n]) = \mathbb{C}_X[n]$), the assertion follows from

$$\mathbb{D} i_s^! \mathbf{R}f_*(\mathbb{C}_X[n]) = i_s^* f_! \mathbb{D}(\mathbb{C}_X[n]) = \mathbf{R}\Gamma_c(X_s, \mathbb{C})[n],$$

where \mathbb{D} denotes the dual in the derived category of bounded complexes of \mathbb{C} -Modules with constructible cohomologies.

1.3. Proposition. *Let $X = \mathbb{A}^n$, and $S = \mathbb{A}^1$. Then \mathcal{G}_f^i has no nontrivial quotient with discrete support, and*

$$(1.3.1) \quad {}^p R^i f_* \mathbb{C}_X = (R^{i-1} f_* \mathbb{C}_X)[1] \quad \text{for any } i.$$

If furthermore f has at most isolated singularities including at infinity [31], then

$$(1.3.2) \quad \tilde{\mathcal{G}}_f^i = 0 \quad \text{for } i \neq 0.$$

Proof. We have a long exact sequence

$$\rightarrow H^{i+n-1}(X, \mathbb{C}) \rightarrow G_f^i \xrightarrow{\partial_t} G_f^i \rightarrow H^{i+n}(X, \mathbb{C}) \rightarrow,$$

because $\mathbf{R}\Gamma(X, \mathbb{C})[n] = C(\partial_t : \mathbf{R}\Gamma(S, \mathcal{K}_f) \rightarrow \mathbf{R}\Gamma(S, \mathcal{K}_f))$. See [4]. This implies the surjectivity of the action of ∂_t on G_f^i , and the first assertion follows.

If f has at most isolated singularities including at infinity (or, more generally, if $\text{supp } \mathcal{E}_s$ in (3.1) has discrete support), then we see that $\varphi_{t_s} {}^p R^{i+n} f_* \mathbb{C}_X = 0$ for $i \neq 0$, using the commutativity of the vanishing cycle functors with the direct image under a proper morphism (because the vanishing cycles form a perverse sheaf with discrete support). This implies that ${}^p R^{i+n} f_* \mathbb{C}_X$ is constant for $i \neq 0$. Then it vanishes for $i \neq 1-n, 0$ by the above exact sequence, and ${}^p R^1 f_* \mathbb{C}_X = \mathbb{C}_S[1]$. So the assertion follows from the Riemann-Hilbert correspondence.

2. Algebraic Brieskorn modules

2.1. Let $f : X \rightarrow S$ and \mathcal{K}_f be as in (1.1). Let $\omega_S \otimes_{\mathcal{O}_S} \mathcal{K}_f$ be the complex of right \mathcal{D}_S -Modules corresponding to the direct image $\mathcal{K}_f (= f_+(\mathcal{O}_X))$. Then by definition of the direct image of right \mathcal{D} -Modules, we have

$$\omega_S \otimes_{\mathcal{O}_S} \mathcal{K}_f = \mathbf{R}f_* \mathcal{C}_f, \quad \omega_S \otimes_{\mathcal{O}_S} \mathcal{G}_f^i = R^i f_* \mathcal{C}_f,$$

where \mathcal{C}_f is the complex of *right* $f^{-1} \mathcal{D}_S$ -Modules such that

$$\mathcal{C}_f^j = \Omega_X^{j+n} \otimes_{f^{-1} \mathcal{O}_S} f^{-1} \mathcal{D}_S,$$

and the differential is given by ${}^r d_f(\omega \otimes P) = d\omega \otimes P + (f^* dt \wedge \omega) \otimes \partial_t P$ for $\omega \in \Omega_X^{j+n}$, $P \in f^{-1} \mathcal{D}_S$, if t is a local coordinate of S and $\partial_t = \partial/\partial t$.

We define the filtration F' on \mathcal{C}_f by $F'_p \mathcal{C}_f^j = 0$ for $p < -1$ and

$$(2.1.1) \quad \begin{aligned} F'_{-1} \mathcal{C}_f^j &= (f^* \Omega_S^1 \wedge \Omega_X^{j+n-1}) \otimes 1, \\ F'_p \mathcal{C}_f^j &= F'_{-1} \mathcal{C}_f^j + \Omega_X^{j+n} \otimes f^{-1} F_{p+j} \mathcal{D}_S, \end{aligned}$$

where F on \mathcal{D}_S is the filtration by the order of operators. (This filtration F' is different from the Hodge filtration F in [27], and is useful only in the isolated singularity case.) Let

$$(2.1.2) \quad \mathcal{C}_f^{(p)} = F'_{-p} \mathcal{C}_f, \quad \mathcal{G}_f^{(p)} = \omega_S^\vee \otimes_{\mathcal{O}_S} R^0 f_* \mathcal{C}_f^{(p)},$$

where ω_S^\vee denotes the dual line bundle of ω_S .

If f is affine and $\dim \text{Sing } f = 0$, then we have a natural morphism $\mathcal{G}_{f,s}^{(0),\text{an}} \rightarrow \mathcal{G}_{f,x}^{(0)}$ for $x \in \text{Sing } X_s$, where $\mathcal{G}_{f,x}^{(0)}$ is the local analytic Brieskorn lattice of f at x [7]. Here $\omega_{S,s}$ is trivialized by using t_s in (1.1). In this case, we define

$$(2.1.3) \quad \mathcal{L}_{f,s}^i = \begin{cases} \text{Ker}(\mathcal{G}_{f,s}^{(0),\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{(0)}) & \text{if } i = 0, \\ \tilde{\mathcal{G}}_{f,s}^{i,\text{an}} & \text{otherwise.} \end{cases}$$

See (1.1) for $\tilde{\mathcal{G}}_f^i$. We define also

$$(2.1.4) \quad \mathcal{G}_f^i = \begin{cases} \mathcal{G}_f^{(0)} & \text{if } i = 0, \\ \mathcal{G}_f^i & \text{otherwise.} \end{cases}$$

2.2. Remarks. (i) If we choose a local coordinate t (by shrinking S), then ω_S is trivialized by dt , and it is well-known that right \mathcal{D}_S -Modules are identified with left \mathcal{D}_S -Modules by using the involution $*$ of \mathcal{D}_S defined by

$$(PQ)^* = Q^*P^*, \quad \partial_t^* = -\partial_t, \quad g^* = g \quad \text{for } g \in \mathcal{O}_S.$$

So we get an isomorphism $\mathcal{K}_f = \mathbf{R}f_*\mathcal{C}_f$ in the derived category of left \mathcal{D}_S -Modules, and

$$\mathcal{C}_f^j = \Omega_X^{j+n} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t],$$

with the differential ${}^l d_f(\omega \otimes \partial_t^j) = d\omega \otimes \partial_t^j - (f^*dt \wedge \omega) \otimes \partial_t^{j+1}$ as is well-known. The action of $f^{-1}\mathcal{D}_S$ on \mathcal{C}_f is given by

$$\partial_t^i(\omega \otimes \partial_t^j) = \omega \otimes \partial_t^{i+j}, \quad g(\omega \otimes 1) = (f^*g)\omega \otimes 1 \quad \text{for } g \in \mathcal{O}_S.$$

(ii) Let $X' = X \setminus \text{Sing } f$. Then, choosing a local coordinate t , we have an isomorphism of complexes

$$(2.2.1) \quad f^*dt \wedge : \Omega_{X'/S}^\bullet[n-1] \rightarrow F'_p\mathcal{C}_f|_{X'} \quad \text{for } p \geq -1.$$

(iii) If f is affine, we have

$$(2.2.2) \quad R^i f_* F'_p \mathcal{C}_f = \mathcal{H}^i(f_* F'_p \mathcal{C}_f).$$

In particular, $R^0 f_* F'_0 \mathcal{C}_f = f_* \Omega_X^n / (f^*dt \wedge df_* \Omega_X^{n-2})$ locally (choosing a local coordinate t), because

$$d(f_*(f^*dt \wedge \Omega_X^{n-2})) = d(f^*dt \wedge f_* \Omega_X^{n-2}) = f^*dt \wedge df_* \Omega_X^{n-2}$$

by $f_*(\text{Im}(f^*dt \wedge : \Omega_X^{n-2} \rightarrow \Omega_X^{n-1})) = \text{Im}(f^*dt \wedge : f_* \Omega_X^{n-2} \rightarrow f_* \Omega_X^{n-1})$.

2.3. Lemma. *With the notation of 2.1, assume X affine and $\dim \text{Sing } f = 0$. Then the natural morphism*

$$(2.3.1) \quad R^i f_* F'_p \mathcal{C}_f \rightarrow \omega_S \otimes_{\mathcal{O}_S} \mathcal{G}_f^i$$

is an isomorphism for $p \geq -1, i \neq 0$, and injective for $i = 0$. In particular, we have

$$(2.3.2) \quad R^i f_* \mathcal{C}_f^{(0)} = \mathcal{G}_f^i \quad \text{for any } i.$$

Proof. Choosing a local coordinate t , $\text{Gr}_p^{F'} \mathcal{C}_f$ is locally isomorphic to $\tau'_{\geq -p}(\Omega_X^\bullet[n], f^*dt \wedge)$ for $p \geq 0$ where $\tau'_{\geq -p}K$ for a complex (K, d) is defined by

$$(\tau'_{\geq -p}K)^i = \text{Coker } d^{p-1} \quad \text{if } i = p, \quad K^i \quad \text{if } i > p, \quad \text{and } 0 \quad \text{otherwise}.$$

Then $\text{Gr}_p^{F'} \mathcal{C}_f$ is quasi-isomorphic to $\Omega_{X/S}^n$ for $p \geq 0$ by hypothesis, and the assertion follows from the long exact sequence

$$\rightarrow R^i f_* F'_{p-1} \mathcal{C}_f \rightarrow R^i f_* F'_p \mathcal{C}_f \rightarrow R^i f_* \text{Gr}_p^{F'} \mathcal{C}_f \rightarrow R^{i+1} f_* F'_{p-1} \mathcal{C}_f \rightarrow$$

because $R^i f_* F'_{p-1} \mathcal{C}_f = 0$ for $i > 0$ by the assumption on f .

2.4. Theorem. *With the notation of (2.1), assume f affine and $\dim \text{Sing } f = 0$. Then we have short exact sequences*

$$(2.4.1) \quad 0 \rightarrow H^1 i_s^! \mathcal{L}_{f,s}^i \rightarrow \tilde{H}^{i+n-1}(X_s, \mathbb{C}) \rightarrow H^0 i_s^! \mathcal{L}_{f,s}^{i+1} \rightarrow 0.$$

Proof. Let $\bar{f} : \bar{X} \rightarrow S$ be a relative compactification of $f : X \rightarrow S$ with $j : X \rightarrow \bar{X}$ the inclusion morphism such that \bar{X} is smooth and $D := \bar{X} \setminus X$ is a divisor. We define $\mathcal{C}_{\bar{f}}^{(0)}$ on \bar{X} as in (2.1). Let $\mathcal{C}_{\bar{f}}^{(0)}(*D)$ be the localization of $\mathcal{C}_{\bar{f}}^{(0)}$ by a local defining equation of D . Then $\mathcal{C}_{\bar{f}}^{(0)}(*D) = j_* \mathcal{C}_f^{(0)}$, and we have a natural surjective morphism

$$\mathcal{C}_{\bar{f}}^{(0)}(*D)^{\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } f} \mathcal{G}_{f,x}^{(0)},$$

where $\mathcal{G}_{f,x}^{(0)}$ is viewed as a sheaf on \bar{X} with support $\{x\}$. Indeed, by the theory of Gauss-Manin connection [7], we have

$$\mathcal{H}^i(\mathcal{C}_f^{(0)}|X_s^{\text{an}}) = \begin{cases} \mathbb{C}\{t_s\} \otimes_{\mathbb{C}} \mathbb{C}_{X_s} & \text{if } i = 1 - n, \\ \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{(0)} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}'_f = \text{Cone}(\mathcal{C}_{\bar{f}}^{(0)}(*D)^{\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } f} \mathcal{G}_{f,x}^{(0)}[-1])$. Then

$$(2.4.2) \quad \mathcal{C}'_f|X_s^{\text{an}} = \mathbb{C}\{t_s\} \otimes_{\mathbb{C}} \mathbb{C}_{X_s}[n-1].$$

Since f is affine and $t_s \mathcal{G}_{f,x}^{(0)} \supset \partial_{t_s}^{-k} \mathcal{G}_{f,x}^{(0)}$ for $k \gg 0$, Nakayama's lemma implies

$$(2.4.3) \quad \mathcal{G}_{f,s}^{(0),\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{(0)} \text{ is surjective.}$$

So we get

$$\mathcal{L}_{f,s}^i = (R^i \bar{f}_* \mathcal{C}'_f)_s \quad \text{for } i \neq 1 - n.$$

For $i = 1 - n$, we have a natural injective morphism $\mathcal{O}_{S^{\text{an}}} \rightarrow R^{1-n} \bar{f}_* \mathcal{C}'_f$, and

$$\mathcal{L}_{f,s}^{1-n} = \text{Coker}(\mathcal{O}_{S^{\text{an}}} \rightarrow R^{1-n} \bar{f}_* \mathcal{C}'_f)_s.$$

Let $i_s^* \mathcal{C}'_f = \text{Cone}(t_s : \mathcal{C}'_f \rightarrow \mathcal{C}'_f)|\bar{X}_s^{\text{an}}$. Let $\bar{j}_s : X_s \rightarrow \bar{X}_s$ denote the inclusion morphism. Then it is enough to show

$$(2.4.4) \quad i_s^* \mathcal{C}'_f = \mathbf{R}(\bar{j}_s)_* \mathbb{C}_X[n-1],$$

using a spectral sequence similar to that in the proof of (1.2). But

$$i_s^* \mathcal{C}'_f|X_s^{\text{an}} = \mathbb{C}_X[n-1]$$

by (2.4.2), and the assertion is reduced to showing a natural quasi-isomorphism

$$(2.4.5) \quad i_s^* \mathcal{C}'_f \xrightarrow{\sim} \mathbf{R}(\bar{j}_s)_* (\bar{j}_s)^* i_s^* \mathcal{C}'_f.$$

Since the assertion is restricted to a neighborhood of D , we may restrict to $\bar{X}' := \bar{X} \setminus \text{Sing } f$, and replace \mathcal{C}'_f with $\mathcal{C}_{\bar{f}}^{(0)}(*D)^{\text{an}}$. Let $X' = X \setminus \text{Sing } f$, $X'_s = X' \cap X_s$ with the inclusion morphisms $j' : X' \rightarrow \bar{X}'$, $j'_s : X'_s \rightarrow \bar{X}'_s$. Then $\mathcal{C}_{\bar{f}}^{(0)}(*D)|_{\bar{X}'} = j'_* \Omega_{X'/S}^\bullet$ by (2.2.1). So (2.4.5) is verified by applying the functor an to the distinguished triangle

$$\rightarrow j'_* \Omega_{X'/S}^\bullet \xrightarrow{t_s} j'_* \Omega_{X'_s/S}^\bullet \rightarrow (j'_s)_* \Omega_{X'_s}^\bullet \rightarrow .$$

This completes the proof of (2.4).

2.5. Corollary. *With the notation of (2.1) and the assumption of (2.4), let*

$$N_s^{i_i} = \dim \text{Ker}(t_s : \mathcal{G}_f^{i_i} \rightarrow \mathcal{G}_f^{i_i}), \quad R_s^{i_i} = \dim \text{Coker}(t_s : \mathcal{G}_f^{i_i} \rightarrow \mathcal{G}_f^{i_i}),$$

for $s \in S$, and let μ_x denote the Milnor number of f at $x \in X$. Then

$$N_s^{i_i+1} + R_s^{i_i} = \dim H^{i+n-1}(X_s, \mathbb{C}) + \delta_{i,0} \sum_{x \in \text{Sing } X_s} \mu_x.$$

Proof. This follows from (2.4) because the local analytic Brieskorn lattice $\mathcal{G}_{f,x}^{(0)}$ is a free $\mathbb{C}\{t\}$ -module of rank μ_x by [30].

2.6. Remark. Let $s \in \Delta$ and $s' \in U$ such that s' is sufficiently near s . Then there does not exist a natural morphism

$$\iota : H^{n-1}(X_s, \mathbb{C}) \rightarrow H^{n-1}(X_{s'}, \mathbb{C})$$

making the following diagram commutative :

$$\begin{array}{ccc} (R^{n-1} f_* \mathbb{C}_X)_s & \xleftarrow{\sim} & H^{n-1}(f^{-1}(D_s), \mathbb{C}) \\ \downarrow & & \downarrow \\ H^{n-1}(X_s, \mathbb{C}) & \xrightarrow{\iota} & H^{n-1}(X_{s'}, \mathbb{C}) \end{array}$$

where the vertical morphisms are natural morphisms, and D_s is a sufficiently small open disk with center s such that $s' \in D_s \setminus \{s\}$. Indeed, the right vertical morphism of the diagram is injective because $(R^{n-1} f_* \mathbb{C}_X)[1]$ is a perverse sheaf. But the left vertical morphism is not bijective if $n = 2$ and $\nu_s \neq 0$ due to Th. 3 of [1]. There exist examples such that $\dim(R^{n-1} f_* \mathbb{C}_X)_s = \dim H^{n-1}(X_s, \mathbb{C})$ with $n = 2$ and $\nu_s \neq 0$ (e.g. $f = x^4 y^2 + 2x^2 y + xy^2$ and $s = -1$).

3. Vanishing cycles

3.1. With the notation of (1.1), let $\bar{f} : \bar{X} \rightarrow S$ be a relative compactification of f with $j : X \rightarrow \bar{X}$ the open immersion such that $\bar{f}j = f$. (Here \bar{f} is proper, but \bar{X} may be singular.) For $s \in \Delta := S \setminus U$, let

$$\mathcal{E}_s = \varphi_{\bar{f}*t_s} \mathbf{R}j_* \mathbb{C}_X[n-1], \quad \mathcal{E}_{s,\infty} = \mathcal{E}_s|_{\bar{X}_s \setminus X_s}, \quad \mathcal{E}_{s,\text{fin}} = \mathcal{E}_s|_{X_s},$$

where φ denotes the vanishing cycle functor [10], and $\bar{X}_s = \bar{f}^{-1}(s)$, etc. We define

$$\begin{aligned} E_s &= \mathbf{H}^0(\bar{X}_s^{\text{an}}, \mathcal{E}_s), & E'_s &= \mathbf{H}^0(\bar{X}_s^{\text{an}}, \mathcal{E}_{s,\infty}), \\ E''_s &= \mathbf{H}^0(X_s^{\text{an}}, \mathcal{E}_{s,\text{fin}}), & E''_x &= (\mathcal{H}^0 \mathcal{E}_s)_x, \end{aligned}$$

for $x \in \text{Sing } X_s$. These vector spaces have naturally the monodromy T associated with the functor φ . Since $\text{supp } \mathcal{E}_{s,\text{fin}} = \text{Sing } X_s$, we have

$$E_s = E'_s \oplus E''_s, \quad E''_s = \bigoplus_{f(x)=s} E''_x.$$

Let $E_s^\lambda = \text{Ker}(T_{ss} - \lambda|_{E_s})$ (similarly for $E_s'^\lambda, E_s''^\lambda$), where T_{ss} is the semisimple part of the monodromy T . Let

$$(3.1.1) \quad \nu_s^\lambda = \dim E_s'^\lambda, \quad \mu_s^\lambda = \dim E_s''^\lambda,$$

and $\nu_s = \sum_\lambda \nu_s^\lambda, \nu = \sum_s \nu_s$ (similarly for μ).

We define $P_s(E, T)$ for $s \in S$ and for a finite dimensional \mathbb{C} -vector space E with a quasi-unipotent automorphism T as follows. Let t_s be as in (1.1), and we will write ∂_t for ∂_{t_s} to simplify the notation. Let $M_s(E, T) = E \otimes_{\mathbb{C}} \mathbb{C}[\partial_t, \partial_t^{-1}]$ with action of ∂_t^i ($i \in \mathbb{Z}$) and t_s defined by

$$\begin{aligned} \partial_t^i(e \otimes \partial_t^j) &= e \otimes \partial_t^{i+j}, \\ t_s(e \otimes \partial_t^j) &= (1-j)e \otimes \partial_t^{j-1} - (2\pi i)^{-1}(\log T)e \otimes \partial_t^{j-1}. \end{aligned}$$

Then $\partial_t^i t_s - t_s \partial_t^i = i \partial_t^{i-1}$ and $t_s \partial_t + j$ on $E \otimes \partial_t^j$ is $-(2\pi i)^{-1} \log T \otimes id$. Here $\log T$ is chosen so that the eigenvalues of $(2\pi i)^{-1} \log T$ are contained in $[0, 1)$. We define

$$\begin{aligned} M_s(E, T)^{>0} &= E \otimes_{\mathbb{C}} \mathbb{C}[\partial_t^{-1}], \\ P_s(E, T) &= M_s(E, T) / M_s(E, T)^{>0} \end{aligned}$$

They are $\mathbb{C}[t_s]$ -modules with action of ∂_t^{-1} .

By a canonical splitting of the filtration V in the one variable case (see [26, 1.5]) together with the commutativity of the de Rham functor with the vanishing cycle functor [18], [20] (see also [25, 3.4.12]), we get canonical isomorphisms

$$(3.1.2) \quad G_f / G_f^{>0} = \bigoplus_{s \in \Delta} P_s(E_s, T), \quad \mathcal{G}_{f,x} / \mathcal{G}_{f,x}^{>0} = P_{f(x)}(E''_x, T),$$

because

$$E_s = {}^p\varphi_{t_s}({}^p R^n f_* \mathbb{C}_X) = {}^p\varphi_{t_s} \text{DR}_S(\mathcal{G}_f^{\text{an}}), \quad E''_x = {}^p\varphi_{t_s} \text{DR}_{S'}(\mathcal{G}_{f,x}),$$

where S' is an analytic open neighborhood of s , and $\mathcal{G}_{f,x}$ denotes also its coherent extension to S' .

3.2. Theorem. *Let $f : X \rightarrow S$ be as in (1.1) and assume X, S affine and $\dim \text{Sing } f = 0$. Then, with the notation of (3.1), we have the commutative diagram of exact sequences in (0.5), where $\mathbb{C}[t], c, t - c$ are replaced respectively by $\Gamma(S, \mathcal{O}_S), s$ and t_s . Furthermore, this holds with $G_f^{(0)}, G_f^{(0), >0}, \mathcal{G}_{f,x}^{(0)}$ replaced by $G_f^{(-1)}, G_f^{(-1), >0}, \mathcal{G}_{f,x}^{(-1)}$. If there exists a nowhere vanishing vector field v on S , and the action of v is bijective on G_f , then the morphisms of the diagram are compatible with the action of v^{-1} .*

Proof. We prove the assertion for $G_f^{(0)}$. The argument is similar for $G_f^{(-1)}$.

The exactness of the middle and right columns follows from (3.1.2). The morphism between the two columns is defined by taking the restriction to the local Milnor fibration at x . So the morphisms of the diagram are naturally defined except for the surjective morphism of the left column, but it is induced by the other morphisms using the commutativity and the exactness.

By GAGA, we may replace $G_f, G_f^{(0)}, G_f^{>0}$ by $\mathcal{G}_f^{\text{an}}, \mathcal{G}_f^{(0), \text{an}}, \mathcal{G}_f^{>0, \text{an}}$ respectively, and then restrict to the stalk at $s \in S$, where $P_s(E_s, T)$, etc. are viewed as sheaves with support $\{s\}$. So it is enough to show the exactness of the middle row and the surjectivity of the second morphism in the upper row, because the bottom row splits, and is exact.

We consider a morphism

$$(3.2.1) \quad \mathcal{G}_{f,s}^{\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x},$$

which is surjective by (2.4.3). The source has the filtration F' defined in (2.1) such that

$$(3.2.2) \quad F'_0 \mathcal{G}_{f,s}^{\text{an}} = \mathcal{G}_{f,s}^{(0), \text{an}}, \quad \text{Gr}_p^{F'} \mathcal{G}_{f,s}^{\text{an}} = (f_* \Omega_{X/S}^n)_s \quad \text{for } p > 0.$$

See (2.2.2). On $\mathcal{G}_{f,x} (= \mathcal{H}^0 \mathcal{C}_{f,x}^{\text{an}})$, F' induces the filtration F' satisfying a similar property where $\mathcal{G}_{f,s}^{\text{an}}, \mathcal{G}_{f,s}^{(0), \text{an}}, (f_* \Omega_{X/S}^n)_s$ are replaced respectively by $\mathcal{G}_{f,x}, \mathcal{G}_{f,x}^{(0)}, \Omega_{X/S,x}^n$. So (3.2.1) induces an isomorphism

$$\text{Gr}_p^{F'} \mathcal{G}_{f,s}^{\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \text{Gr}_p^{F'} \mathcal{G}_{f,x} \quad \text{for } p > 0,$$

and hence $\mathcal{G}_{f,s}^{\text{an}} / \mathcal{G}_{f,s}^{(0), \text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x} / \mathcal{G}_{f,x}^{(0)}$ is an isomorphism. This shows the exactness of the middle row.

Now it remains to show the surjectivity of

$$\mathcal{G}_{f,s}^{>0, \text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{>0}.$$

But (3.2.1) is surjective, and the assertion follows from the property of the Deligne extension (or the filtration V). See for example [24, 3.1.5].

The last assertion on the action on v^{-1} is clear.

3.3. Remark. Assume f is affine, and has at most isolated singularities including at infinity. Let $G_f^{(0)}$ be as in (2.1). As a corollary of the results of Sabbah [33], f is cohomologically

tame in his sense if and only if $G_f^{(0)}$ is coherent over \mathcal{O}_S . For the “if” part it is sufficient to show $G_f^{(0)} \subset G_f^{>0}$, and this is easily verified by using the filtration of Kashiwara [18] and Malgrange [20]. See loc. cit. For the converse, it is sufficient, however, to note that, for a good filtration F of a (regular) holonomic \mathcal{D}_S -Module M , we have for each $s \in S$

$$(3.3.1) \quad \dim \mathrm{Gr}_p^F M_s = \dim {}^p\varphi_{t_s} \mathrm{DR}_S(M) \quad \text{for } p \gg 0.$$

Indeed, if $(M, F) = (\mathcal{G}_f, F')$, we have $\dim \mathrm{Gr}_p^F M_s = \mu_s$ for $p \gg 0$, and the above equality gives $\nu_s = 0$. For the proof of (3.3.1), note that $\dim \mathrm{Gr}_p^F M_s$ for $p \gg 0$ coincides with the multiplicity, and is independent of the choice of the good filtration F . (In the regular holonomic case, we can take $F_p = V^{\alpha-p}$ ($p > 0$) for some $\alpha \in \mathbb{Q}$, where V denotes the filtration of Kashiwara [18] and Malgrange [20] in the one variable case.)

3.4. Proof of Corollary (0.7). Let S be a smooth analytic curve, and Δ a discrete subset. Put $U = S \setminus \Delta$. Let L be a \mathbb{C} -local system on U with quasi-unipotent local monodromies around any $s \in \Delta$, and \mathcal{L} the meromorphic Deligne extension. That is, \mathcal{L} is a regular holonomic \mathcal{D}_S -Module such that $\mathcal{L}|_U = L \otimes_{\mathbb{C}} \mathcal{O}_U$ and for any $s \in \Delta$, the action of local equation of $\{s\}$ is bijective on \mathcal{L}_s . Let $\mathcal{L}^{(0)}$ be a coherent \mathcal{O}_S -submodule of \mathcal{L} such that $\mathcal{L}^{(0)}|_U = \mathcal{L}|_U$. We define $\mathrm{ord}_s \mathcal{L}^{(0)}$, the order of $\mathcal{L}^{(0)}$ at $s \in \Delta$, as follows.

Let S' be a sufficiently small open disk around s with a coordinate t , and $\{\omega_i\}$ an $\mathcal{O}_{S'}$ -basis of $\mathcal{L}|_{S'}$. Let L^* be the dual local system of L , and $\{e_j\}$ a basis of horizontal (multivalued) sections of $L^*|_{S' \setminus \{s\}}$. Then $\langle e_j, \omega_i \rangle$ is a Nilson class function on $S' \setminus \{s\}$, and the determinant has the asymptotic expansion

$$\det \langle e_j, \omega_i \rangle = Ct^\alpha + \text{higher terms}$$

for $C \in \mathbb{C}^*$ and $\alpha \in \mathbb{Q}$, because $\det L$ has local monodromies with finite order. This α is independent of the choice of the bases, and we define $\mathrm{ord}_s \mathcal{L}^{(0)} = \alpha$. Then we get (0.7) from the following observations.

(i) If L has a finite filtration G , then it induces the filtration G on \mathcal{L} and $\mathcal{L}^{(0)}$, and

$$\mathrm{ord}_s \mathcal{L}^{(0)} = \prod_j \mathrm{ord}_s \mathrm{Gr}_j^G \mathcal{L}^{(0)}.$$

(ii) If $\mathcal{L}^{(0)}$ is the Deligne extension such that the eigenvalues of the residues are contained in $(\alpha, \alpha + 1]$ (resp. $[\alpha, \alpha + 1)$), then

$$\mathrm{ord}_s \mathcal{L}^{(0)} = \sum_{j=1}^r \alpha_j,$$

where $r = \mathrm{rank} L$, and the α_j are rational numbers contained in $(\alpha, \alpha + 1]$ (resp. $[\alpha, \alpha + 1)$) such that $\exp(-2\pi i \alpha_j)$ are the eigenvalues of the local monodromy of L around s (with multiplicity).

(iii) If $\mathcal{L}^{(0)}$ is a coherent extension of the local analytic Brieskorn lattice $\mathcal{G}_{f,x}^{(0)}$ (resp. $\mathcal{G}_{f,x}^{(-1)}$), then

$$\text{ord}_s \mathcal{L}^{(0)} = (n-2)\mu_x/2 \quad (\text{resp. } n\mu_x/2).$$

This is well-known after [36].

(iv) If $\mathcal{L}^{(0)}$ is $\mathcal{G}_{f,s}^{(0),>0,\text{an}}$ with the assumption of (3.2), then

$$\text{ord}_s \mathcal{L}^{(0)} = (-\nu_s^{\neq 1} + \sum_{x \in \text{Sing } X_s} (n-2)\mu_x)/2.$$

where $\nu_s^{\neq 1} = \sum_{\lambda \neq 1} \nu_s^\lambda$. This follows from the above observations together with the short exact sequence

$$0 \rightarrow \text{Kernel} \rightarrow \mathcal{G}_{f,s}^{(0),>0,\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{(0)} \rightarrow 0,$$

where $\text{Kernel} = \text{Ker}(\mathcal{G}_{f,s}^{>0,\text{an}} \rightarrow \bigoplus_{x \in \text{Sing } X_s} \mathcal{G}_{f,x}^{>0})$ by (3.2). (The argument is similar for $\mathcal{G}_{f,s}^{(-1),>0,\text{an}}$ with $(n-2)\mu_x$ replaced by $n\mu_x$.)

(v) If L and e_j are defined over \mathbb{Z} , then $\det\langle e_j, \omega_i \rangle^2$ is a meromorphic function on S . If furthermore S , \mathcal{L} , $\mathcal{L}^{(0)}$ and ω_i are algebraic, and \mathcal{L} is also regular at infinity, then $\det\langle e_j, \omega_i \rangle^2$ is a rational function on S .

3.5. Proposition. *With the notation and assumption of (3.2), let $F'_p G_f = \partial_t^p G_f^{(0)}$, $F_p \mathcal{G}_{f,x} = \partial_t^p \mathcal{G}_{f,x}^{(0)}$, and let V_s denote the filtration by the eigenvalue of the action of $\partial_{t_s} t_s$ indexed by \mathbb{Q} (i.e. $\partial_{t_s} t_s - \alpha$ is nilpotent on $\text{Gr}_{V_s}^\alpha$). See [9], [18], [20], etc. Then*

$$\text{Gr}_p^{F'} \text{Gr}_{V_s}^\alpha G_f = \bigoplus_{x \in \text{Sing } X_s} \text{Gr}_p^F \text{Gr}_{V_s}^\alpha \mathcal{G}_{f,x}$$

for $p \geq 0$ and $0 < \alpha \leq 1$. In particular, $G_f^{(0)}$ together with the filtration V_s gives the sums of the Hodge numbers of the local Milnor fibers.

Proof. Let F' denote also the corresponding filtration on \mathcal{G}_f , $\mathcal{G}_f^{\text{an}}$. Then we have a canonical isomorphism

$$\text{Gr}_p^{F'} \text{Gr}_{V_s}^\alpha G_f = \text{Gr}_p^{F'} \text{Gr}_{V_s}^\alpha \mathcal{G}_{f,s}^{\text{an}}.$$

by the exactness of the functors involved. We have furthermore

$$\text{Gr}_p^{F'} \mathcal{G}_{f,s}^{\text{an}} = \bigoplus_{x \in \text{Sing } X_s} \text{Gr}_p^F \mathcal{G}_{f,x},$$

because $\text{Gr}_p^{F'} G_f = \bigoplus_s \text{Gr}_p^{F'} \mathcal{G}_{f,s}^{\text{an}}$. So it is enough to show the bistrict surjectivity of

$$r_s : (\mathcal{G}_{f,s}^{\text{an}}; F', V_s) \rightarrow \bigoplus_{x \in \text{Sing } X_s} (\mathcal{G}_{f,x}; F, V_s),$$

where the filtration F' is restricted to the index $p \geq -1$. Since the strict surjectivity is separately clear, the assertion is reduced to the compatibility of three submodules $\text{Ker } r_s$, F'_p , V_s^α of $\mathcal{G}_{f,s}^{\text{an}}$ due to [24, 1.2.14]. But this is clear because $\text{Ker } r_s \subset F'_p$ for $p \geq -1$ by (3.2). The last assertion follows from [35] (see also [22], [27], [29], etc.)

Remark. We can also show for $\alpha \neq 0$

$$\dim \text{Gr}_{V_s}^{\alpha+1} G_f^{(0)} - \dim \text{Gr}_{V_s}^\alpha G_f^{(0)} = \sum_{x \in \text{Sing } X_s} (\dim \text{Gr}_{V_s}^{\alpha+1} \mathcal{G}_{f,x}^{(0)} - \dim \text{Gr}_{V_s}^\alpha \mathcal{G}_{f,x}^{(0)}),$$

because $\dim \text{Gr}_{V_s}^{\alpha+1} \mathcal{L}_{f,s} = \dim \text{Gr}_{V_s}^\alpha \mathcal{L}_{f,s}$ for $\alpha \neq 0$ with the notation of (0.4). This is closer to Varchenko's formulation [35]. However, it gives only the Hodge numbers $\dim \text{Gr}_F^p H^{n-1}(X_x, \mathbb{C})_\lambda$ for $p \neq n-1$ or $\lambda \neq 1$ (because $\alpha \neq 0$), where X_x denotes the local Milnor fiber at $x \in \text{Sing } X_s$, and the index λ means the dimension of the λ -eigenspace by the monodromy. To get the Hodge number for $p = n-1$ and $\lambda = 1$, we have to use also $G_f^{(-1)}$.

3.6. Relation between the numerical invariants. Let $X = \mathbb{A}^n$, and $S = \mathbb{A}^1$. Assume f has at most isolated singularities including at infinity [31] (or, more generally, the $\text{supp } \mathcal{E}_s$ are discrete.) Let μ_s, ν_s, μ, ν be as in (3.1), and $m = \text{rank } L|_U (= \dim H^{n-1}(X_s, \mathbb{C})$ for a general s). Then we can easily show that the vanishing of $H^i(S^{\text{an}}, L)$ for any i implies

$$(3.6.1) \quad m = \mu + \nu,$$

which was first obtained by [31], [33]. Here apparently different definitions of ν are used, but they are all equivalent, because we can also show

$$(3.6.2) \quad \chi(X_s) - \chi(X_{s'}) = (-1)^n (\nu_s + \mu_s)$$

for $s, s' \in S$ such that $s' \notin \Delta$, and the corresponding formula is proved in loc. cit.

For $s \in S$, let $\rho_s^\lambda, \rho_s'^\lambda$ be the number of Jordan blocks of T (i.e. the minimal number of generators over $\mathbb{C}[T]$) on $E_s^\lambda, E_s'^\lambda$ respectively. We define

$$\begin{aligned} R'_s &= \dim \text{Coker}(t_s : G_f^{(0)} \rightarrow G_f^{(0)}), & R_s &= \dim \text{Coker}(t_s : G_f \rightarrow G_f), \\ N'_s &= \dim \text{Ker}(t_s : G_f^{(0)} \rightarrow G_f^{(0)}), & N_s &= \dim \text{Ker}(t_s : G_f \rightarrow G_f). \end{aligned}$$

Let T_s denote the local monodromy of the local system $L|_U (= R^{n-1} f_* \mathbb{C}_X|_U)$ around $s \in \Delta$. Let $\tilde{h}^i(X_s) = \dim \tilde{H}^i(X_s, \mathbb{C})$, $\tilde{h}_c^i(X_s) = \dim \tilde{H}_c^i(X_s, \mathbb{C})$. Then the relation between $\tilde{h}^i(X_s)$, $\tilde{h}_c^i(X_s)$, R'_s , R_s , N'_s , N_s , ν_s , μ_s , μ_x , $\rho_s'^1$, ρ_s^1 , T_s and m is summarized as follows.

$$\begin{aligned} \tilde{h}^i(X_s) &= 0 \text{ if } i \neq n-2, n-1, & \tilde{h}_c^i(X_s) &= 0 \text{ if } i \neq n-1, n, \\ R'_s &= \tilde{h}^{n-1}(X_s) + \sum_{x \in \text{Sing } X_s} \mu_x, & R_s &= \tilde{h}_c^{n-1}(X_s) = \dim \text{Coker}(T_s - id), \\ N'_s &= \tilde{h}^{n-2}(X_s) \leq \rho_s'^1, & N_s &= \tilde{h}_c^n(X_s) \leq \rho_s^1, \\ R'_s - N'_s &= m - \nu_s, & R_s - N_s &= m - \mu_s - \nu_s. \end{aligned}$$

In particular, we can calculate the dimension of the cohomology (with compact support) of any fiber using the action of t on $G_f^{(0)}, G_f$ together with the Milnor numbers μ_x . Most of the relations follow from (1.2–3), (2.4) and (3.6.1). The formula involving $\text{Coker}(T_s - id)$ is closely related to Th. 1 of [1], and the vanishing of the Betti numbers to [34]. See also [12], [13], [14], [31], [33], etc.

In the case $n = 2$, we can show that the monodromy T on $E_s^1 := {}^p\varphi_{t_s,1}L[1]$ is semisimple so that $\rho_s^1 = \mu_s^1 + \nu_s^1$, $\rho_s'^1 = \nu_s^1$, where ${}^p\varphi_{t_s,1}$ is the unipotent monodromy part of ${}^p\varphi_{t_s}$.

3.7. Remark. In the case $X = \mathbb{A}^n, S = \mathbb{A}^1$, we can calculate the Gauss-Manin system and the Brieskorn-Deligne lattice in the following way provided that the polynomial f is not very complicated :

- (i) Calculate $df \wedge d\Omega^{n-2}$ to get a basis of $G_f^{(0)} = \Omega^n/df \wedge d\Omega^{n-2}$ over \mathbb{C} .
- (ii) Calculate the action of t on the basis to get generators of $G_f^{(0)}$ over $\mathbb{C}[t]$.
- (iii) Calculate ∂_t^{-1} (using $\partial_t^{-1}(\omega) = df \wedge \omega'$ for $d\omega' = \omega$) to get the differential equations and determine $G_f^{(0), >0}$.

Here $\Omega^p = \Gamma(X, \Omega_X^p)$ as in the introduction. The argument is similar to [7] except that our case is algebraic and global. For example, we use ∂_t^{-1} rather than ∂_t , and $(\prod_c (t - c)^{k_c})\partial_t$ is used at the last stage only for a direct factor or a subquotient of G_f .

3.8. Examples. (i) $f = y^2 + x^3 - 3x$. This is the simplest example such that $\bar{L} \neq 0$ in the notation of (0.8). We have $h = g = 1$, $m = 2$, $r = \nu = 0$, $\mu = 2$, $\mu_{\pm 2}^1 = 1$, $R'_{\pm 2} = 2$, $R_{\pm 2} = 1$, $N'_{\pm 2} = N_{\pm 2} = 0$, and get the Gauss hypergeometric differential equation. The calculation seems much easier than the conventional one using the discriminant and the integration by parts as explained in the introduction of [21]. The indicial equation [8] is compatible with $G_f^{(0)} = G_f^{>0}$.

(ii) $f = x^4y^2 + 2x^2y + xy^2$. This is an example such that $h'' (= \dim H^1(S^{\text{an}}, j_*\bar{L})) \neq 0$. We have $m = 7$, $g = 1$, $h = 5$, $r = 1$, $h' = h'' = 2$, $\mu = \mu_0 = 4$, $\nu = \nu_{-1} = 3$, (where ν can be calculated as in [15] combined with [19]), and $R'_0 = 7$, $R_0 = 5$, $N'_0 = 0$, $N_0 = 2$, $R'_{-1} = R_{-1} = 4$, $N'_{-1} = N_{-1} = 0$.

(iii) $f = x^2y^2 + 2xy + x$. This is an example such that $\nu \neq 0$ and $\bigcap_i \partial_t^{-i} G_f^{(0)} = 0$. We have $h = m = 2$, $h' = r = 1$, $g = 0$, $\mu = \mu_0^1 = 1$, $\nu = \nu_{-1}^{-1} = 1$, and $\mathcal{G}_f = \mathcal{D}_S/\mathcal{D}_{St}\partial_t((t+1)\partial_t + 1/2)$. In this case $G_f^{(0)} \rightarrow G_f/\text{tor}G_f$ is bijective.

Note that the condition $\mu > 0$ does not necessarily imply that $\bigcap_{i \in \mathbb{N}} \partial_t^{-i} G_f^{(0)} = 0$ (e.g. $f = x^2y^2 + x^2 + 2x$ or $f = x^6y^3 + x^2y^2 + 2xy$). This property holds for the local analytic Brieskorn lattice, and hence it is true if we take a completion of \mathcal{O} by some topology associated with $x \in \text{Sing } f$. But it is not true without taking it.

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